On consistency of noncommutative Chern-Simons theory

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ABSTRACT: We consider the noncommutative extension of Chern-Simons theory. We show the the theory can be fully expanded in power series of the noncommutative parameter θ and that no non-analytical sector exists. The theory appears to be unstable under radiative corrections, but we show that the infinite set of instabilities, to all orders in $\hbar$ and in θ, is confined to a BRS exact cocycle. We show also that the theory is anomaly free. The quantum theory cannot be written in terms of the Groenewald-Moyal star product, and hence doubts arise on the interpretation of the noncommutative nature of the underlying spacetime. Nonetheless, the deformed theory is well defined as a quantum field theory, and the beta function of the Chern-Simons coupling constant vanishes, as in the ordinary Chern-Simons theory.

Contents

1. Introduction

It is common wisdom that a noncommutative extension of a quantum field theory can be realized by replacing the standard commutative product with the Groenewald-Moyal star product [1, 2]. When applied to gauge theories, this approach should worry about the fate of all the symmetries which, in the commutative case, are taken as the very definition of the model and which allow a complete discussion of its renormalizability by analyzing the integrated cohomology of the BRS operator in the ghost charge zero sector (stability) and ghost charge one sector (anomaly). We would like to assume this point of view from the very beginning and consider a noncommutative quantum gauge field model to be defined by its symmetries, locality and power counting. Of course the presence of the $\theta_{\mu\nu}$ parameter with inverse square mass dimension deeply affects the results of the cohomology analysis; in a previous investigation [3, 4], where we have applied this method to the two dimensional BF model whose action is expanded in power series of θ, we found it to be unstable already at first order in θ and also showed that the Groenewald-Moyal extension is not the general solution of the symmetry constraints. In this paper we discuss in the same framework the noncommutative Chern-Simons model, whose action has a necessarily analytic expansion in the θ parameter. We are able to carry out the analysis to all orders and the Chern-Simons model turns out to be much more robust than the BF one with respect to noncommutative deformations. Indeed, the model is unstable in the sense that to any fixed order in θ, the classical action acquires new contributions with new free parameters, but these contributions never belong to the BRS cohomology, *i.e.* are BRS variations. Even more important, the theory stays anomaly free to all orders. Now, the absence of anomaly implies that the symmetry is maintained in the full noncommutative extension and the fact that the new contributions to the action are BRS trivial leads to the

consideration that the noncommutative model can still be regarded as "renormalizable" in a wider sense [5, 6], since all new parameters belong to the nonphysical sector of the theory. Needless to say, the Groenewald-Moyal star product does not coincide with the general extension we propose here. The plan of the paper is as follows: in section 2 we briefly recall the defining symmetries of the classical model, the noncommutative Groenewald-Moyal extension and set up the tools to analyze the quantum theory to all orders in θ. Section 3 is devoted to the explicit discussion of the stability and anomaly problem up to the second order in θ; the computation, whose feasibility relies on a previous result on the general solution of the linear vector supersymmetry [7], gives us a hint of what might be the all order result, which is proven in detail in the appendix. Our conclusive considerations are collected in section 4.

2. The classical model

The ordinary, commutative, Chern-Simons theory reads

$$S_{\text{CS}} = \frac{k}{2} \, \text{Tr} \int d^3x \; \epsilon^{\mu\nu\rho} \left(\mathbf{A}_\mu \partial_\nu \mathbf{A}_\rho - i\frac{2}{3} \mathbf{A}_\mu \mathbf{A}_\nu \mathbf{A}_\rho \right) , \tag{2.1}$$

where $\mathbf{A}_\mu \equiv T^a A_\mu^a$ and the trace must be done on the group generators, which, for SU(n), obey

$$\text{Tr}\,(T^a T^b) = \delta^{ab} \tag{2.2}$$

$$\left[T^a, T^b \right] = i f^{abc} T^c \tag{2.3}$$

$$\{T^a, T^b\} = d^{abc} T^c + \frac{1}{n} \delta^{ab} \tag{2.4}$$

The action (2.1) is invariant under the nilpotent BRS transformations

$$\begin{aligned}
sA_\mu^a &= -(D_\mu c)^a \equiv -(\partial_\mu c^a + f^{abc} A_\mu^b c^c) \\
sc^a &= +\frac{1}{2} f^{abc} c^b c^c \\
s\bar{c}^a &= b^a \\
sb^a &= 0 \,,
\end{aligned} \tag{2.5}$$

where the fields $c^a(x)$, $\bar{c}^a(x)$ and $b^a(x)$ represent ghost, antighost and Lagrange multiplier, respectively, and belong to the adjoint representation of the gauge group.

The gauge fixing term is

$$\begin{aligned}
S_{gf} &= s \int d^3x \; \bar{c}^a \partial^\mu A_\mu^a \\
&= \int d^3x \; (b^a \partial A^a + \bar{c}^a \partial^\mu (D_\mu c)^a) \; .
\end{aligned} \tag{2.6}$$

Notice that, in three dimensions the gauge parameter being massive, the Landau gauge choice is mandatory.

Once gauge fixed, the action

$$S = S_{\text{CS}} + S_{gf} \tag{2.7}$$

is invariant under an additional vector symmetry [8]

$$\delta_\mu S = 0 \,, \tag{2.8}$$

where

$$
\begin{aligned}
\delta_\mu A_\nu^a &= \frac{1}{k} \epsilon_{\mu\nu\rho} \partial^\rho \bar{c}^a \\
\delta_\mu c^a &= -A_\mu^a \\
\delta_\mu \bar{c}^a &= 0 \\
\delta_\mu b^a &= \partial_\mu \bar{c}^a \;.
\end{aligned}
\tag{2.9}
$$

The vector symmetry (2.9) is peculiar to all topological field theories, and the algebra formed with the BRS operator is

$$s^2 = 0 \tag{2.10}$$

$$\{\delta_\mu, \delta_\nu\} = 0 \tag{2.11}$$

$$\{s, \delta_\mu\} = \partial_\mu + \text{eqs of motion.} \tag{2.12}$$

This algebraic structure, like the ordinary global supersymmetry, closes on translations, and plays a crucial role in the proof of finiteness of Chern-Simons theory, and of topological quantum field theories in general [9, 10].

Besides BRS symmetry (2.5) and supersymmetry (2.9), the action (2.7) shares with all gauge field theories built in the Landau gauge, the ghost equation [11]

$$\int d^3x \, \left(\frac{\delta}{\delta c^a} + f^{abc} \bar{c}^b \frac{\delta}{\delta b^c} \right) S \equiv \mathcal{G}^a S = 0 \;. \tag{2.13}$$

In order to proceed towards the noncommutative extension of Chern-Simons theory, it is customary to deform the ordinary product between quantum fields, into the Groenewald - Moyal "star" product [1, 2]

$$\phi(x)\psi(x) \longrightarrow \phi(x) * \psi(x) \equiv \lim_{y \to x} \, \exp(\frac{i}{2}\theta^{\mu\nu}\partial_\mu^x \partial_\nu^y) \, \phi(x)\psi(y) \,, \tag{2.14}$$

where $\theta^{\mu\nu}$ is a rank-two antisymmetric matrix which controls the noncommutative nature of spacetime coordinates

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} \;. \tag{2.15}$$

Consequently, the noncommutative Chern-Simons (NCCS) action reads

$$S_{\text{NCCS}} = \frac{k}{2} \, \text{Tr} \, \int d^3x \, \epsilon^{\mu\nu\rho} \left(\mathbf{A}_\mu * \partial_\nu \mathbf{A}_\rho - i\frac{2}{3}\mathbf{A}_\mu * \mathbf{A}_\nu * \mathbf{A}_\rho \right) \;. \tag{2.16}$$

The action (2.16) can be fully expanded in power series of θ

$$S_{\text{NCCS}} = \sum_{n=0}^{\infty} S_{\text{NCCS}}^{(n)} \,, \tag{2.17}$$

where

$$S_{\text{NCCS}}^{(0)} = S \tag{2.18}$$

$$S_{\text{NCCS}}^{(1)} = \theta^{\alpha\beta} S_{\alpha\beta} \tag{2.19}$$

$$S_{\text{NCCS}}^{(2)} = \theta^{\alpha\beta}\theta^{\gamma\delta} S_{\alpha\beta\gamma\delta}\,, \tag{2.20}$$

and so on at higher orders. Up to second order in θ, S is given by eq. (2.7), and

$$S_{\alpha\beta} = \frac{1}{2}d^{abc} \int d^3x \; \partial_\alpha A_\mu^a \left(\frac{1}{6}\varepsilon^{\mu\nu\rho}\partial_\beta A_\nu^b A_\rho^c - \partial^\mu \bar{c}^b \partial_\beta c^c \right) \tag{2.21}$$

$$S_{\alpha\beta\gamma\delta} = -\frac{1}{8}f^{abc} \int d^3x \; \partial_{\alpha\gamma} A_\mu^a \left(\frac{1}{6}\varepsilon^{\mu\nu\rho}\partial_{\beta\delta} A_\nu^b A_\rho^c + \partial^\mu \bar{c}^b \partial_{\beta\delta} c^c \right) \tag{2.22}$$

Accordingly, always up to $O(\theta^2)$, the noncommutative BRS symmetry is

$$s^{(\theta)} A_\mu^a = sA_\mu^a - \frac{1}{2}\theta^{\alpha\beta}d^{abc}\partial_\alpha A_\mu^b \partial_\beta c^c + \frac{1}{8}\theta^{\alpha\beta}\theta^{\gamma\delta}f^{abc}\partial_{\alpha\gamma}A_\mu^b \partial_{\beta\delta}c^c$$

$$s^{(\theta)} c^a = sc^a + \frac{1}{4}\theta^{\alpha\beta}d^{abc}\partial_\alpha c^b \partial_\beta c^c - \frac{1}{16}\theta^{\alpha\beta}\theta^{\gamma\delta}f^{abc}\partial_{\alpha\gamma}c^b \partial_{\beta\delta}c^c \tag{2.23}$$

$$s^{(\theta)} \bar{c}^a = s\bar{c}^a$$

$$s^{(\theta)} b^a = sb^a\,.$$

On the other hand, the supersymmetry δ_μ, being linear in the quantum fields, is not affected by the noncommutative extension

$$\delta_\mu^{(\theta)} = \delta_\mu\,. \tag{2.24}$$

As a nontrivial property, it can be verified that the noncommutative deformation shares with the ordinary theory the symmetries

$$s^{(\theta)} S_{\text{NCCS}} = 0 \tag{2.25}$$

$$\delta_\mu S_{\text{NCCS}} = 0 \tag{2.26}$$

$$\mathcal{G}^a S_{\text{NCCS}} = 0 \tag{2.27}$$

and the algebraic structure

$$\left(s^{(\theta)} \right)^2 = 0 \tag{2.28}$$

$$\{\delta_\mu, \delta_\nu\} = 0 \tag{2.29}$$

$$\{s^{(\theta)}, \delta_\mu\} = \partial_\mu + \text{eqs of motion}\,. \tag{2.30}$$

Two remarks are in order.

The first concerns the choice of the gauge group of noncommutative gauge field theories, which is known that it should be U(n) [1, 2]. The reason, is that the gauge group U(n) is closed under the star product while SU(n), for instance, is not. This restriction does not reveal itself in the noncommutative action, because of the trace which is done on the group generators, which are traceless. But it is evident when composite operators are considered,

like $s^{(\theta)}\mathbf{A}_\mu$ and $s^{(\theta)}\mathbf{c}$ in the BRS transformations. These expressions, indeed, involve the anticommutator (2.4), which does not form an algebra, due to the central term $\frac{1}{n}\delta^{ab}$, and are meaningful only for U(n) gauge groups, for which the central term disappears

$$\text{U}(n) \ : \ \{T^a, T^b\} = d^{abc}T^c \ . \tag{2.31}$$

More subtle is the necessity of U(n) gauge groups to verify, for instance, the nilpotency of the noncommutative BRS operator $s^{(\theta)}$ (2.28), order by order in θ. Nilpotency is achieved only thanks to the following nontrivial relation between structure constants and completely symmetric tensors

$$f^{abp}f^{cdp} = d^{acp}d^{bdp} - d^{bcp}d^{adp} \tag{2.32}$$

which holds for U(n) groups only [12].

The second remark concerns the analyticity of the theory. The quantum action Γ_{NCCS} does not have any non-analytic sector in θ. Indeed, let us suppose that the NCCS quantum action Γ_{NCCS} contains a sector which can be expanded in negative powers of θ

$$\Gamma_{\text{NCCS}}|_{non\ analytic} = \sum_{n=1}^{\infty} \frac{1}{\theta^n}\Gamma_{\text{NCCS}}^{(n)} \ . \tag{2.33}$$

recalling that $\frac{1}{\theta}$ has mass dimensions $+2$, power counting implies that the mass dimension of $\Gamma_{\text{NCCS}}^{(n)}$ is

$$\dim\left(\Gamma_{\text{NCCS}}^{(n)}\right) = 3 - 2n \,, \tag{2.34}$$

which, of course, must be a non-negative quantity. Thus, at most there is only one possible term in the non-analytic expansion (2.33)

$$\Gamma_{\text{NCCS}}|_{non\ analytic} = \frac{1}{\theta}\Gamma_{\text{NCCS}}^{(1)} \,, \tag{2.35}$$

but no such $\Gamma_{\text{NCCS}}^{(1)}$, with mass dimension $+1$, can be constructed which is a color singlet and gauge invariant.

Hence, the θ-expansion of NCCS theory does not admit a non-analytical sector.

3. The quantum extension

Once we got rid of the non-analytical sector, the counterterm $\Sigma_c^{(\theta)}$ can be fully expanded in power series of θ:

$$\Sigma_c^{(\theta)} = \sum_{n=0}^{\infty} \theta^n \cdot \Sigma_c^{(n)} \tag{3.1}$$

In (3.1), the "dot" product denotes all possible ways to contract Lorentz indices in order to form a scalar quantity. For example

$$\theta \cdot \Sigma_c^{(1)} \equiv \theta_{\mu\nu}\Sigma^{(1)\mu\nu} \tag{3.2}$$

$$\theta^2 \cdot \Sigma_c^{(2)} \equiv \theta_{\mu\nu}\theta^{\mu\nu}\Sigma^{(2)} + \theta_{\mu\lambda}\theta_\nu^\lambda\Sigma^{(2)\mu\nu} + \theta_{\mu\nu}\theta_{\rho\sigma}\Sigma^{\mu\nu\rho\sigma} \tag{3.3}$$

In order that the action is stable under radiative corrections, the counterterm must obey the following constraints[1]

$$\mathcal{G}^a \Sigma_c^{(\theta)} = 0 \tag{3.4}$$

$$s^{(\theta)} \Sigma_c^{(\theta)} = 0 \tag{3.5}$$

$$\delta_\mu \Sigma_c^{(\theta)} = 0 \tag{3.6}$$

Recalling that the noncommutative BRS operator can be θ-expanded

$$s^{(\theta)} = s^{(0)} + s^{(1)} + s^{(2)} + \cdots \tag{3.7}$$

order by order in θ, the stability equation (3.5) reads, up to $O(\theta^2)$

$$O(\theta^0) : s^{(0)} \Sigma_c^{(0)} = 0 \tag{3.8}$$

$$O(\theta^1) : s^{(0)} \Sigma_c^{(1)} + s^{(1)} \Sigma_c^{(0)} = 0 \tag{3.9}$$

$$O(\theta^2) : s^{(0)} \Sigma_c^{(2)} + s^{(1)} \Sigma_c^{(1)} + s^{(2)} \Sigma_c^{(0)} = 0 \tag{3.10}$$

while the ghost equation (3.4) and the supersymmetry constraint (3.6), which do not mix the θ-sectors, hold at each order

$$\mathcal{G}^a \Sigma_c^{(n)} = 0 \tag{3.11}$$

$$\delta_\mu \Sigma_c^{(n)} = 0 . \tag{3.12}$$

For the analysis of the quantum extension of the theory, it is extremely helpful to know the general solution of the supersymmetry equation

$$\delta_\mu X_q^p = 0 , \tag{3.13}$$

where p and q denote respectively mass dimension and ghost number of the functional X. In [7], it has been proven that the most general solution of (3.13) is

$$X_q^p = \varepsilon^{\mu\nu\rho} \delta_\mu \delta_\nu \delta_\rho X_{q+3}^{p-3} \equiv \delta^3 X_{q+3}^{p-3} . \tag{3.14}$$

Now, $\Sigma_c^{(0)}$ is a local integrated functional with mass dimensions $+3$ and ghost number 0. According to eq. (3.12) and eq. (3.14), it must be written

$$\Sigma_c^{(0)} = \delta^3 X_3^0 . \tag{3.15}$$

Since the ghost operator $\mathcal{G}^a$ (2.13) anticommutes with both the BRS and the supersymmetry operators

$$\{\mathcal{G}^a, s^{(\theta)}\} = \{\mathcal{G}^a, \delta_\mu\} = 0 , \tag{3.16}$$

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[1] We omit to introduce external fields to define the nonlinear BRS variations in (2.23). It is readily seen indeed, that their presence, not altering at all our results, would make the treatment much heavier. Therefore, without loss of generality, also at the quantum level we shall continue to deal with the BRS operator, and not with the Slavnov-Taylor identity [13].

it must also be

$$\mathcal{G}^a X_3^0 = 0 \,, \tag{3.17}$$

but no functional with the correct quantum numbers exists, hence

$$X_3^0 = 0 \,, \tag{3.18}$$

and

$$\Sigma_c^{(0)} = 0 \,. \tag{3.19}$$

We recovered here in a few lines a result which is already known, concerning the finiteness of commutative Chern-Simons theory [14, 15]. This same technique easily leads us to get new results at higher orders in θ.

Taking into account (3.19), at the first order in θ, $\Sigma_c^{(1)}$ must obey

$$s^{(0)}\Sigma_c^{(1)} = 0 \tag{3.20}$$

$$\mathcal{G}^a \Sigma_c^{(1)} = 0 \tag{3.21}$$

$$\delta_\alpha \Sigma_c^{(1)} = 0 \tag{3.22}$$

From eq. (3.22), we have

$$\Sigma_c^{(1)} = \delta^3 \Sigma_{\Phi\Pi=3}^{dim=2} \,, \tag{3.23}$$

since, in order that $\theta \cdot \Sigma_c^{(1)}$ has mass dimension $+3$, $\Sigma_c^{(1)}$ must have dimension $+5$. In general

$$\dim(\Sigma_c^{(n)}) = 3 + 2n \,. \tag{3.24}$$

The only possible term, satisfying also the ghost condition (3.21), is

$$\Sigma_{3\mu\nu}^2 = \int d^3x \; d^{abc} c^a \partial_\mu c^b \partial_\nu c^c \,, \tag{3.25}$$

which is not BRS invariant

$$s^{(0)}\Sigma_{3\mu\nu}^2 \neq 0 \,, \tag{3.26}$$

and therefore also

$$s^{(0)}\delta\Sigma_{\mu\nu}^{(1)} \neq 0 \,, \tag{3.27}$$

since, on integrated functionals,

$$\{s^{(0)}, \delta\} = 0 \,. \tag{3.28}$$

Hence, $\Sigma_{\mu\nu}^{(1)}$ is ruled out by the symmetry constraints, and

$$\Sigma_c^{(1)} = 0 \,. \tag{3.29}$$

Therefore, the NCCS theory, at least at first order in θ, not only is stable under radiative corrections, but, more than that, keeps the property of finiteness displayed by the commutative theory.

At the next order in θ, taking into account the previous results $\Sigma_c^{(0)} = \Sigma_c^{(1)} = 0$, the constraints on the counterterm become

$$s^{(0)}\Sigma_c^{(2)} = 0 \qquad (3.30)$$

$$\mathcal{G}^a \Sigma_c^{(2)} = 0 \qquad (3.31)$$

$$\delta_\mu \Sigma_c^{(2)} = 0 \qquad (3.32)$$

Again, the most general solution of the supersymmetry condition (3.32), is

$$\Sigma_c^{(2)} = \delta^3 \Sigma_{\Phi\Pi=3}^{dim=4} \ . \qquad (3.33)$$

The situation here is a bit more involved: the most general $O(\theta^2)$ candidate satisfying the ghost equation (3.31) and the supersymmetry condition (3.32), turns out to be

$$\theta^2 \cdot \Sigma_c^{(2)} = \delta^3 \left(\theta^2 \cdot \Sigma_3^4\right) = \delta^3 \int d^3x \ \left(\theta_{\mu\nu}\theta^{\mu\nu}\Sigma + \theta^{\mu\lambda}\theta_\lambda^\nu \Sigma_{\mu\nu} + \theta^{\mu\nu}\theta^{\rho\sigma}\Sigma_{\mu\nu\rho\sigma}\right) \qquad (3.34)$$

where

$$\Sigma = \int d^3x \ \left(T_1^{[ab]cd} c^{a\rho} c_\rho^b c^{c\sigma} A_\sigma^d + \alpha_1 f^{abc} c^{a\rho\sigma} c_\rho^b c_\sigma^c\right) \qquad (3.35)$$

$$\Sigma_{\mu\nu} = \int d^3x \ \left(T_2^{[ab]cd} c_\mu^a c_\nu^b c^{c\sigma} A_\sigma^d + T_3^{[ab]cd} c^{a\rho} c_\rho^b c_\mu^c A_\nu^d + \right.$$
$$\left. + (\alpha_2 d^{abc} + \alpha_3 f^{abc}) c_{\mu\sigma}^a c_\nu^b c^{c\sigma}\right) \qquad (3.36)$$

$$\Sigma_{\mu\nu\rho\sigma} = \int d^3x \ \left(T_4^{(ab)cd} c_\mu^a c_\nu^b c_\rho^c A_\sigma^d + \alpha_4 f^{abc} c_{\mu\rho}^a c_\nu^b c_\sigma^c\right) \ , \qquad (3.37)$$

where $c_\mu^a(x) \equiv \partial_\mu c^a(x)$, α_i are constants and T_i^{abcd} are constant invariant tensors. Square and round brackets mean antisymmetrization and symmetrization of color indices, respectively.

The $s^{(0)}$ operator, which does not depend on θ, does not mix the three sectors which form $\theta^2 \cdot \Sigma_c^{(2)}$, hence each of them can be studied separately

$$s^{(0)}\Sigma = s^{(0)}\Sigma_{\mu\nu} = s^{(0)}\Sigma_{\mu\nu\rho\sigma} = 0 \qquad (3.38)$$

where again we used eq. (3.28).

A careful analysis of the above BRS conditions leads to the following relations between the free parameters

sector $\theta_{\mu\nu}\theta^{\mu\nu}$

$$T_1^{[ab]cd} - T_1^{[ab]dc} = \alpha_1 f^{abp} f^{pcd} \qquad (3.39)$$

$$T_1^{[ab]cd} + T_1^{[ab]dc} \equiv T_1^{[ab](cd)} \ : \ \text{undetermined} \qquad (3.40)$$

sector $\theta_\mu^\lambda \theta_{\nu\lambda}$

$$T_2^{[ab][cd]} + T_3^{[cd][ab]} - \frac{\alpha_2}{4}\left(d^{pbd}f^{pac} - d^{pad}f^{pbc} - d^{pbc}f^{pad} + d^{pac}f^{pbd}\right)$$
$$-\frac{\alpha_3}{2}f^{abp}f^{pcd} = 0 \qquad (3.41)$$

$$T_2^{[ab](cd)} , \ T_3^{[ab](cd)} \ : \ \text{undetermined} \qquad (3.42)$$

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sector $\theta_{\mu\nu}\theta_{\rho\sigma}$

$$T_4^{(ab)cd} + T_4^{(ab)dc} = 3\alpha_4 d^{abp} d^{pcd} \tag{3.43}$$

$$T_4^{(ab)cd} - T_4^{(ab)dc} \equiv T_4^{(ab)[cd]} \ : \ \text{undetermined} . \tag{3.44}$$

Notice that, in order to write eq. (3.43), we used the relation (2.32), which holds only for $U(n)$ groups.

The three terms which form the counterterm (3.34) are then

$$\Sigma = \int d^3x \left(\left(\frac{\alpha_1}{2} f^{abp} f^{pcd} + T_1^{[ab](cd)} \right) c^{a\rho} c_\rho^b c^{c\sigma} A_\sigma^d + \alpha_1 f^{abc} c^{a\rho} c_\rho^b c_\sigma^c \right) \tag{3.45}$$

$$\Sigma_{\mu\nu} = \int d^3x \left(T_2^{[ab]cd} c_\mu^a c_\nu^b c^{c\sigma} A_\sigma^d + T_3^{[ab]cd} c^{a\rho} c_\rho^b c_\mu^c A_\nu^d + (\alpha_2 d^{abc} + \alpha_3 f^{abc}) c_{\mu\sigma}^a c_\nu^b c^{c\sigma} \right) \tag{3.46}$$

$$\Sigma_{\mu\nu\rho\sigma} = \int d^3x \left(\left(\frac{3}{2}\alpha_4 d^{abp} d^{pcd} + T_4^{(ab)[cd]} \right) c_\mu^a c_\nu^b c_\rho^c A_\sigma^d + \alpha_4 f^{abc} c_{\mu\rho}^a c_\nu^b c_\sigma^c \right) . \tag{3.47}$$

The noncommutative theory, hence, starting from order $O(\theta^2)$, not only breaks finiteness, but it is not even stable under radiative corrections, as the counterterm cannot be reabsorbed by a renormalization of field and parameter of the classical theory, nor it can be expressed in terms of the Groenewald-Moyal product.

Nonetheless, we verified that, at least at $O(\theta^2)$, the above nine-fold (as many are the free parameters) instability belongs to the nonphysical sector of the theory, since the unstable counterterm can be written as an exact BRS cocycle

$$\theta^2 \cdot \Sigma_c^{(2)} = s^{(0)} \delta^3 \int d^3x \ \left(\theta_{\mu\nu}\theta^{\mu\nu}\widehat{\Sigma} + \theta^{\mu\lambda}\theta_\lambda^\nu \widehat{\Sigma}_{\mu\nu} + \theta^{\mu\nu}\theta^{\rho\sigma}\widehat{\Sigma}_{\mu\nu\rho\sigma} \right) \tag{3.48}$$

where

$$\widehat{\Sigma} = \int d^3x \ \left(\frac{\alpha_1}{2} f^{abc} c^{a\rho} c_\rho^b \partial A^c - \frac{1}{2} T_1^{[ab](cd)} c^{a\rho} c_\rho^b A^{c\sigma} A_\sigma^d \right) \tag{3.49}$$

$$\widehat{\Sigma}_{\mu\nu} = - \int d^3x \ \left((\alpha_2 d^{abc} + \alpha_3 f^{abc}) A^{a\rho} c_{\mu\rho}^b c_\nu^c - T_3^{[ab][cd]} c^{a\rho} A_\rho^b c_\mu^c A_\nu^d \right. \tag{3.50}$$

$$\left. + \left(\frac{\alpha_2}{4} f^{abp} d^{pcd} + \frac{1}{2} T_2^{[ab](cd)} \right) c_\mu^a c_\nu^b A^{c\rho} A_\rho^d + \frac{1}{2} T_3^{[ab](cd)} c^{a\rho} c_\rho^b A_\mu^c A_\nu^d \right)$$

$$\widehat{\Sigma}_{\mu\nu\rho\sigma} = \int d^3x \ \left(\alpha_4 f^{abc} c_\nu^a c_\rho^b \partial_\mu A_\sigma^c - \frac{1}{2} T_4^{(ab)[cd]} c_\mu^a c_\nu^b A_\rho^c A_\sigma^d \right) \tag{3.51}$$

Notice that, in order to write the counterterm as an exact BRS cocycle, we had to use the relation (3.41), which, although not attractive, turns out to be important to get this result.

Hence, we found that, till order $O(\theta^2)$, the counterterm can be written as

$$\Sigma_c^{(2)} = s^{(\theta)} \delta^3 \Sigma_{\Phi\Pi=2}^{dim=0} \Big|_{O(\theta^2)} , \tag{3.52}$$

where Σ_2^0 is a generic power series in θ, which depends on the ghost field $c^a(x)$ only if differentiated. In the appendix, we show that the integrated cohomology of the BRS

operator $s^{(\theta)}$ in the ghost sectors 0 and $+1$ is trivial, and hence the most general solution of the BRS constraint (3.5) is

$$\Sigma_c^{(\theta)} = s^{(\theta)} \widehat{\Sigma} \ . \tag{3.53}$$

This result, together with eq.(3.14) on the solution of the supersymmetry condition, leads us to conclude that, *to all orders in $\hbar$ and θ*, the most general counterterm of the NCCS theory is indeed of the form (3.52)

$$\Sigma_c^{(\theta)} = s^{(\theta)} \delta^3 \Sigma_{\Phi\Pi=2}^{dim=0} \ . \tag{3.54}$$

We shall comment on this in the Conclusions.

For what concerns anomaly, analogous results hold. The noncommutative anomaly $\mathcal{A}^{(\theta)}$, which has mass dimensions $+3$ and ghost number $+1$, must satisfy the same constraints as the counterterm

$$\mathcal{G}^a \mathcal{A}^{(\theta)} = 0 \tag{3.55}$$

$$\delta_\mu \mathcal{A}^{(\theta)} = 0 \tag{3.56}$$

$$s^{(\theta)} \mathcal{A}^{(\theta)} = \left(\sum_{n=0}^{\infty} \theta^n \cdot s^{(n)} \right) \left(\sum_{m=0}^{\infty} \theta^m \cdot \mathcal{A}^{(m)} \right) = 0 \, , \tag{3.57}$$

and $\mathcal{A}^{(\theta)}$ must be closed but not exact

$$\mathcal{A}^{(\theta)} \neq s^{(\theta)} \widehat{\mathcal{A}} \ . \tag{3.58}$$

The first two conditions (3.55) and (3.56) are solved by

$$\mathcal{A}^{(\theta)} = \delta^3 \ \mathcal{A}_{\Phi\Pi=4}^{dim=0} \, , \tag{3.59}$$

and $\mathcal{A}_4^0$ is a local integrated functional depending on the ghost field only if differentiated.

The BRS condition (3.57), up to second order in θ, reads

$$O(\theta^0) : s^{(0)} \mathcal{A}^{(0)} = 0 \tag{3.60}$$

$$O(\theta^1) : s^{(0)} \mathcal{A}^{(1)} + s^{(1)} \mathcal{A}^{(0)} = 0 \tag{3.61}$$

$$O(\theta^2) : s^{(0)} \mathcal{A}^{(2)} + s^{(1)} \mathcal{A}^{(1)} + s^{(2)} \mathcal{A}^{(0)} = 0 \ . \tag{3.62}$$

Now, $\mathcal{A}^{(0)}$ and $\mathcal{A}^{(1)}$ are ruled out by the solution (3.59): the commutative theory and its noncommutative extension at $O(\theta)$ are not anomalous

$$\mathcal{A}^{(0)} = \mathcal{A}^{(1)} = 0 \ . \tag{3.63}$$

At $O(\theta^2)$ the most general solution of the constraints is

$$A^{(2)} = \delta^3 \int d^3x \ \left(T_1^{[ab][cd]} \theta_{\mu\nu} \theta^{\mu\nu} c^{a\rho} c_\rho^b c^{c\sigma} c_\sigma^d + T_2^{[ab][cd]} \theta_\mu^\lambda \theta_{\lambda\nu} c^{a\mu} c^{b\nu} c^{c\rho} c_\rho^d + \right.$$
$$\left. + T_3^{[ab][cd]} \theta_{\mu\nu} \theta_{\rho\sigma} c^{a\mu} c^{b\nu} c^{c\rho} c^{d\sigma} \right) \, , \tag{3.64}$$

which can be expresses as an exact BRS cocycle

$$A^{(2)} = s^{(0)} \delta^3 \int d^3x \ \left(T_1^{[ab][cd]} \theta_{\mu\nu} \theta^{\mu\nu} A^{a\rho} c_\rho^b c^{c\sigma} c_\sigma^d + T_2^{[ab][cd]} \theta_\mu^\lambda \theta_{\lambda\nu} A^{a\mu} c^{b\nu} c^{c\rho} c_\rho^d + \right.$$
$$\left. + T_3^{[ab][cd]} \theta_{\mu\nu} \theta_{\rho\sigma} A^{a\mu} c^{b\nu} c^{c\rho} c^{d\sigma} \right) , \tag{3.65}$$

and therefore, also at order $O(\theta)^2$, we explicitly checked that the noncommutative theory is not anomalous.

In the appendix we show that this result holds to all orders in θ (and $\hbar$) as well

$$\mathcal{A}^{(\theta)} = 0 . \tag{3.66}$$

4. Conclusions

In this paper we considered the noncommutative Chern-Simons theory, expanded in the noncommutativity parameter $\theta_{\mu\nu}$. This expansion covers the whole theory, since we showed that a non-analytical expansion is not allowed in this case. Therefore, we faced with a double expansion: a quantum expansion in $\hbar$ and a noncommutative one in $\theta_{\mu\nu}$.

We gave the most general expression for the counterterm in (3.54). Due to the presence of a generic functional Σ_2^0, the counterterm depends on an infinite number of free parameters, and, consequently, represents an infinite set of unstable radiative corrections to the classical noncommutative action. Moreover, the counterterm cannot be written in terms of the Groenewald-Moyal star product, which therefore turns out to be unstable under radiative corrections. This seems to indicate that the quantum theory loses its link to an underlying noncommutative structure of spacetime.

The optimistic counterpart, is that all the above considerations are confined to the nonphysical sector of the quantum theory, which is also anomaly free. The bulk of the theory maintains unaltered the good properties of the commutative one: the β function of the noncommutative Chern-Simons coupling constant vanishes. We proved also that the noncommutative parameter θ is a nonphysical coupling constant, since, like what happens for gauge parameters, the fact that the counterterm is an exact BRS cocycle implies that

$$\theta_{\mu\nu} \frac{\partial \Gamma^{(\theta)}}{\partial \theta_{\mu\nu}} = s^{(\theta)} \int d^3x \ \Delta \cdot \Gamma^{(\theta)} , \tag{4.1}$$

where $\Gamma^{(\theta)}$ is the quantum noncommutative action, and $\Delta \cdot \Gamma^{(\theta)}$ is a quantum insertion.

These results suggest to infer that the noncommutative extension leads to a quantum field theory which is consistent as long as the physical sector is concerned, while it is less meaningful for the part of the theory which determines the anomalous dimensions.

We stress that the job has been greatly simplified in the case we considered: a three dimensional topological field theory whose dependence on the noncommutative parameter is completely analytic, and whose set of symmetries, in particular the vector supersymmetry, allowed us to study thoroughly higher orders in both $\hbar$ and θ.

It would be extremely interesting to make analogous investigations in more physically relevant quantum field theories, like for instance Yang-Mills, or also Maxwell theory.

A. All orders results

A.1 Anomaly

We want to show that the general solution of the Wess-Zumino consistency condition on the anomaly

$$s^{(\theta)}\mathcal{A}^{(\theta)} = 0 \qquad (A.1)$$

is an exact BRS cocycle

$$\mathcal{A}^{(\theta)} = s^{(\theta)}\widehat{\mathcal{A}}^{(\theta)} . \qquad (A.2)$$

Where $\mathcal{A}^\theta$ and $\widehat{\mathcal{A}}^\theta$ are integrated local functionals with mass dimension $+3$ and ghost number $+1$, which can be expressed, like the BRS operator $s^{(\theta)}$, as power series in θ

$$\mathcal{A}^{(\theta)} = \sum_{n=0}^{\infty} \theta^n \cdot \mathcal{A}^{(n)} \qquad (A.3)$$

$$s^{(\theta)} = \sum_{n=0}^{\infty} \theta^n \cdot s^{(n)} , \qquad (A.4)$$

where the same prescription as in (3.1) is adopted concerning the "dot" products.

The global equation (A.1), written on local forms, reads

$$s^{(\theta)}\mathcal{A}_1^3(x) + d\mathcal{A}_2^2(x) = 0 , \qquad (A.5)$$

where $\mathcal{A}_q^p(x)$ is a local p-form with ghost number q, and d is the exterior derivative.

In section 3 we showed that, up to second order in θ, it holds

$$(\mathcal{A}_1^3)^{(2)} = \left(s^{(\theta)}\mathcal{A}_0^3 + d\mathcal{A}_1^2\right)^{(2)} = s^{(0)}(\mathcal{A}_0^3)^{(2)} + s^{(1)}(\mathcal{A}_0^3)^{(1)} + s^{(2)}(\mathcal{A}_0^3)^{(0)} + d(\mathcal{A}_1^2)^{(2)} , \quad (A.6)$$

and we neglect that, in particular, $(\mathcal{A}_0^3)^{(0)} = (\mathcal{A}_0^3)^{(1)} = 0$, due to the ghost equation.

The proof develops by induction: we assume that, up to order $n-1$, it holds

$$\begin{aligned}(\mathcal{A}_1^3)^{(n-1)} &= \left(s^{(\theta)}\mathcal{A}_0^3 + d\mathcal{A}_1^2\right)^{(n-1)} \\ &= s^{(0)}(\mathcal{A}_0^3)^{(n-1)} + s^{(1)}(\mathcal{A}_0^3)^{(n-2)} + \cdots + s^{(n-2)}(\mathcal{A}_0^3)^{(1)} \\ &\quad + s^{(n-1)}(\mathcal{A}_0^3)^{(0)} + d(\mathcal{A}_1^2)^{(n-1)} , \end{aligned} \qquad (A.7)$$

and we want to show that, also at the next order,

$$\begin{aligned}(\mathcal{A}_1^3)^{(n)} &= \left(s^{(\theta)}\mathcal{A}_0^3 + d\mathcal{A}_1^2\right)^{(n)} \\ &= s^{(0)}(\mathcal{A}_0^3)^{(n)} + s^{(1)}(\mathcal{A}_0^3)^{(n-1)} + \cdots + s^{(n-1)}(\mathcal{A}_0^3)^{(1)} \\ &\quad + s^{(n)}(\mathcal{A}_0^3)^{(0)} + d(\mathcal{A}_1^2)^{(n)} , \end{aligned} \qquad (A.8)$$

where $(\mathcal{A}_1^3)^{(n)}$ satisfies the equation (A.5) at the order $O(\theta^n)$, that is

$$s^{(0)}(\mathcal{A}_1^3)^{(n)} + s^{(1)}(\mathcal{A}_1^3)^{(n-1)} + \cdots + s^{(n-1)}(\mathcal{A}_1^3)^{(1)} + s^{(n)}(\mathcal{A}_1^3)^{(0)} + d(\mathcal{A}_2^2)^{(n)} = 0 . \qquad (A.9)$$

Substituting (A.7) in (A.9), and grouping the same θ-powers of $\mathcal{A}_0^3$, we get

$$
\begin{aligned}
s^{(0)}(\mathcal{A}_1^3)^{(n)}+ \\
s^{(1)}s^{(0)}(\mathcal{A}_0^3)^{(n-1)}+ \\
((s^{(1)})^2 + s^{(2)}s^{(0)})(\mathcal{A}_0^3)^{(n-2)} + \cdots + \\
(s^{(1)}s^{(n-2)} + s^{(2)}s^{(n-3)} + \cdots + s^{(n-2)}s^{(1)} + s^{(n-1)}s^{(0)})(\mathcal{A}_0^3)^{(1)}+ \\
(s^{(1)}s^{(n-1)} + s^{(2)}s^{(n-2)} + \cdots + s^{(n-1)}s^{(1)} + s^{(n)}s^{(0)})(\mathcal{A}_0^3)^{(0)}+ \\
d(\widehat{\mathcal{A}}_2^2)^{(n)} = 0 \ .
\end{aligned} \tag{A.10}
$$

Using the nilpotency relation at $O(\theta^n)$

$$
s^{(0)}s^{(n)} + s^{(1)}s^{(n-1)} + \cdots + s^{(n-1)}s^{(1)} + s^{(n)}s^{(0)} = 0 , \tag{A.11}
$$

the eq.(A.10) writes

$$
s^{(0)}\left((\mathcal{A}_1^3)^{(n)} - s^{(1)}(\mathcal{A}_0^3)^{(n-1)} - s^{(2)}(\mathcal{A}_0^3)^{(n-2)} - \cdots - s^{(n-1)}(\mathcal{A}_0^3)^{(1)} - \right.
$$
$$
\left. s^{(n)}(\mathcal{A}_0^3)^{(0)}\right) + d(\widehat{\mathcal{A}}_2^2)^{(n)} = 0 , \quad \text{(A.12)}
$$

which is an equation of cohomology modulo-d for the ordinary, commutative, nilpotent, BRS operator $s^{(0)}$.

At this point, we proceed as usual, applying $s^{(0)}$ to both sides of (A.12), using the anticommutation relation $\{s^{(0)}, d\} = 0$, the nilpotency $d^2 = 0$, and the fact that the cohomology of d is empty, and we get the following descent equations [13], which hold for each $O(\theta^n)$

$$
s^{(0)}(\widehat{\mathcal{A}}_2^2)^{(n)} + d(\mathcal{A}_3^1)^{(n)} = 0 \tag{A.13}
$$
$$
s^{(0)}(\mathcal{A}_3^1)^{(n)} + d(\mathcal{A}_4^0)^{(n)} = 0 \tag{A.14}
$$
$$
s^{(0)}(\mathcal{A}_4^0)^{(n)} = 0 \ . \tag{A.15}
$$

The last equation (A.15) is a local cohomology equation for $s^{(0)}$, and we know that it is formed by odd polynomials in the undifferentiated ghost $c^a(x)$ [13]

$$
\mathcal{H}(s^{(0)}) = \mathcal{P}_{\text{odd}}(c) \ . \tag{A.16}
$$

Therefore, in the sector with even ghost number, the local cohomology of $s^{(0)}$ is empty, and the solution of (A.15) is

$$
(\mathcal{A}_4^0)^{(n)} = s^{(0)}(\mathcal{A}_3^0)^{(n)} \ . \tag{A.17}
$$

Now, it is easy to mount the descent equations, remembering (A.16) and the fact that the ghost field has vanishing mass dimensions. We get

$$
(\widehat{\mathcal{A}}_2^2)^{(n)} = s^{(0)}(\mathcal{A}_1^2)^{(n)} + d(\mathcal{A}_2^1)^{(n)} , \tag{A.18}
$$

which, substituted in (A.12), transforms the problem of cohomology modulo-d into a problem of local cohomology

$$
s^{(0)}\left((\mathcal{A}_1^3)^{(n)} - s^{(1)}(\mathcal{A}_0^3)^{(n-1)} - \cdots - s^{(n-1)}(\mathcal{A}_0^3)^{(1)} - s^{(n)}(\mathcal{A}_0^3)^{(0)} - d(\mathcal{A}_1^2)^{(n)}\right) = 0 , \quad \text{(A.19)}
$$

which is solved by (A.8), which was our aim.

Therefore, at any order in θ, the solution of the Wess-Zumino consistency condition (A.1) is only the trivial one (A.2), and the noncommutative Chern-Simons theory is not anomalous.

A.2 Counterterm

The computation of the counterterm follows the same steps as the anomaly. We want to solve the BRS constraint on the counterterm

$$s^{(\theta)} \Sigma^{(\theta)} = 0 \,, \tag{A.20}$$

where $\Sigma^{(\theta)}$ is a local integrated functional with mass dimensions $+3$ and ghost number 0, which can be expanded in power series of θ

$$\Sigma^{(\theta)} = \sum_{n=0}^{\infty} \theta^n \cdot \Sigma^{(n)} \,. \tag{A.21}$$

Written in terms of differential forms, (A.20) reads

$$s^{(\theta)} \Sigma_0^3(x) + d\Sigma_1^2(x) = 0 \,. \tag{A.22}$$

In section 3 we showed that, up to second order in θ,

$$(\Sigma_0^3)^{(2)} = (s^\theta \Sigma_{-1}^3 + d\Sigma_0^2)^{(2)} \,. \tag{A.23}$$

By induction, we suppose that, up to $O(\theta^{(n-1)})$,

$$(\Sigma_0^3)^{(n-1)} = \left(s^{(\theta)} \Sigma_{-1}^3 + d\Sigma_0^2 \right)^{(n-1)} \,, \tag{A.24}$$

and we want to show that, at the next order,

$$(\Sigma_0^3)^{(n)} = \left(s^{(\theta)} \Sigma_{-1}^3 + d\Sigma_0^2 \right)^{(n)} \,, \tag{A.25}$$

where $(\Sigma_0^3)^{(n)}$ is constrained to satisfy (A.22). Substituting (A.24) in (A.22), reordering terms and using the nilpotency relation (A.11), in an analogous way to the anomaly case, we land on a local cohomology modulo-d problem for the ordinary BRS operator $s^{(0)}$:

$$s^{(0)} \left((\Sigma_0^3)^{(n)} - s^{(1)} (\Sigma_{-1}^3)^{(n-1)} - \cdots - s^{(n)} (\Sigma_{-1}^3)^{(0)} \right) + d(\widehat{\Sigma}_1^2)^{(n)} = 0 \,, \tag{A.26}$$

which yields the descent equations

$$s^{(0)} (\widehat{\Sigma}_1^2)^{(n)} + d(\Sigma_2^1)^{(n)} = 0 \tag{A.27}$$

$$s^{(0)} (\Sigma_2^1)^{(n)} + d(\Sigma_3^0)^{(n)} = 0 \tag{A.28}$$

$$s^{(0)} (\Sigma_3^0)^{(n)} = 0 \,. \tag{A.29}$$

The last (A.29) is a local cohomology equation for the ordinary BRS operator, for which the result (A.16) holds. This time the ghost charge of the 0-form $(\Sigma_3^0)^{(n)}$ is odd, but,

since its mass dimension is $2n$, the local cohomology also in this case is vanishing, since no dimensionful quantities can be constructed with polynomials of undifferentiated ghost fields. Hence

$$(\Sigma_3^0)^{(n)} = s^{(0)}(\Sigma_2^0)^{(n)} . \tag{A.30}$$

The descent equations are easily mounted, up to

$$(\widehat{\Sigma}_1^2)^{(n)} = s^{(0)}(\Sigma_0^2)^{(n)} + d(\Sigma_1^1)^{(n)} , \tag{A.31}$$

which, substituted in (A.26), leads to a local cohomology problem, solved by (A.25).

Therefore, we have shown that the integrated cohomology of $s^{(\theta)}$ is empty in all sectors $O(\theta^n)$, with $n \geq 1$.

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